

Online Appendix to “Modeling Persistent Interest Rates with Double-Autoregressive Processes”

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Main paper is available [here](#)

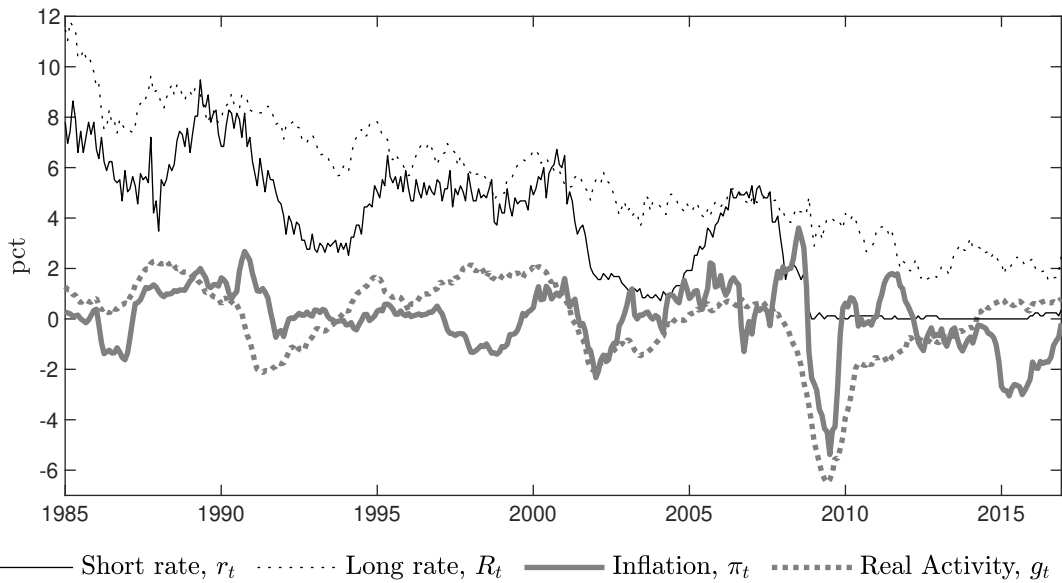
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B Data

The interest rate data and the macroeconomic factors constructed as explained in the main paper are exhibited in Figure 1. Table 1 details how the macroeconomic factors, π_t and g_t , correlate with the underlying observed data as well as the fractions of total variation in these data that they capture. The inflation factor, π_t , is highly correlated with both inflation measurements and explains 85 pct of the total variation in these data. The real activity factor, g_t , correlates most strongly with the employment growth rate and the HELP index. Our measure of real activity captures 66 pct of the variation in these underlying data.

Figure 1: Data on Interest Rates and Macroeconomic Factors



Note: The interest rates, r_t and R_t , are the 1-month Treasury bill rate and the 10-year Treasury bond yield. The inflation factor, π_t , is the first principal component of growth rates in the CPI and PPI indices. The real activity factor, g_t , is the first principal component of the unemployment rate, the employment growth rate, and the industrial production and HELP indices. The data are measured at the monthly frequency.

Table 1: Interpretation of the Macroeconomic Factors

Explained (pct)		Empirical Correlation Coefficients					
		CPI	PPI	UNEMP	EMP	PROD	HELP
π_t	85.49	0.92	0.92	-	-	-	-
g_t	65.75	-	-	-0.71	0.93	0.71	0.87

Note: The left column shows the percentage variation that the principal components π_t and g_t explain of the underlying data. The right columns show the empirical correlation coefficients between π_t and g_t and the raw data series given by growth rates in the CPI and PPI indices, the unemployment rate (UNEMP), the employment growth rate (EMP), the industrial production index (PROD), and the HELP index.

C Estimation Results

C.1 Specification of the DAR Model

We specify the model in (6) in the main paper by use of conventional methods. Specifically, the rank is determined by the likelihood-ratio test in Johansen (1991), which is based on the linear VAR model, with critical values obtained using the wild bootstrap procedure in Cavaliere, Rahbek, and Taylor (2014). The test model is specified with a restricted constant, i.e., a constant term appears in the cointegrating relations only.¹ The lag structure of the test is specified by general-to-specific LR tests, information criteria, and misspecification tests. For the choice of lags up to 3 months, the residuals are not autocorrelated according to univariate Ljung-Box tests, see the right panel of Table 4 in the paper. Likelihood-ratio tests, for which results are reported in Table 2, suggest a reduced rank of $r = 2$. We interpret these findings as indications that the DAR model may be well-specified by $r = 2$ and lag length $K = 3$. Results from misspecification testing in the paper confirm this presumption.

We estimate the DAR model with $r = 2$ and $K = 3$ by maximum likelihood under just-identifying restrictions. Then, we impose further restrictions to obtain models with economically sensible interpretations. The restrictions are imposed sequentially starting

¹If a constant is unrestricted, the CVAR model implies that the data contains a linear trend (Johansen, 1995).

Table 2: Rank Testing

$r \leq 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
90.89	32.62	15.99	5.06
[0.00]	[0.03]	[0.07]	[0.11]

Note: Likelihood-ratio test of the null $r \leq \{0, 1, 2, 3\}$ against $r = p$. P-values obtained with the wild bootstrap in [Cavaliere, Rahbek, and Taylor \(2014\)](#) in brackets.

with setting the most insignificant estimate to zero first in the relations $\beta'X_t$ and then in the adjustment matrix α . At each step, the restrictions are tested by likelihood-ratio tests.

C.2 Parameter Estimates

Parameter estimates are reported for the DAR, CVAR, and stationary VAR models in Tables 3–5. The CVAR model with rank $r = 2$ is estimated using the same procedure as described for the DAR model above. The stationary VAR model is econometrically identified and is therefore estimated without imposing restrictions. The CVAR and stationary VAR models are estimated with the same lag length as the DAR model.

Table 3: Parameter Estimates of the DAR model

$\mu' (\times 10^{-4})$				$\mu^{\mathbb{Q}'} (\times 10^{-4})$			
-0.041 (0.060)	0.288 (0.217)	-0.273 (0.263)	-0.244 (0.106)	-0.637 (0.448)	-0.215 (0.119)	-0.122 (0.425)	-0.241 (0.095)
α'				β'			
0	0	0.017 (0.008)	0.008 (0.003)	1	0	-3.672 (1.150)	-1.681 (0.611)
0	-0.031 (0.021)	0	0	-1	1	0	0
Γ_1				Γ_2			
-0.175 (0.064)	0.059 (0.034)	0.027 (0.022)	0.036 (0.039)	-0.028 (0.072)	-0.032 (0.023)	0.011 (0.019)	0.014 (0.045)
-0.026 (0.034)	0.027 (0.055)	0.150 (0.050)	0.140 (0.102)	-0.035 (0.040)	-0.119 (0.055)	0.064 (0.044)	0.048 (0.106)
0.039 (0.032)	0.032 (0.056)	0.444 (0.059)	-0.087 (0.113)	0.036 (0.032)	-0.063 (0.049)	-0.162 (0.066)	0.292 (0.118)
0.027 (0.016)	0.010 (0.027)	0.068 (0.022)	0.244 (0.048)	0.008 (0.017)	-0.015 (0.025)	0.025 (0.029)	0.187 (0.053)
Γ_3				$\Phi^{\mathbb{Q}}$			
-0.061 (0.056)	-0.022 (0.022)	-0.036 (0.0220)	0.061 (0.038)	0.758 (0.018)	0.242 (0.000)	0.093 (0.090)	0.225 (0.078)
-0.045 (0.034)	0.065 (0.058)	-0.090 (0.043)	0.001 (0.096)	-0.008 (0.016)	1.007 (0.000)	-0.001 (0.082)	-0.003 (0.070)
0.007 (0.029)	0.064 (0.051)	0.122 (0.056)	-0.084 (0.113)	0.886 (0.010)	-0.756 (0.000)	0.477 (0.049)	-0.887 (0.043)
-0.039 (0.015)	-0.015 (0.025)	0.034 (0.025)	0.288 (0.052)	0.807 (0.019)	-0.738 (0.000)	-0.438 (0.093)	0.138 (0.080)
$\Sigma_0(\times 10^{-3})$				Σ_1			
0.045 (0.010)				-0.096 (0.013)	-0.019 (0.006)	-0.015 (0.006)	-0.008 (0.005)
0.049 (0.037)	0.217 (0.012)			-0.015 (0.014)	0.007 (0.010)	0.012 (0.011)	-0.028 (0.012)
-0.016 (0.040)	0.022 (0.016)	0.247 (0.017)		-0.006 (0.018)	-0.007 (0.011)	-0.012 (0.018)	0.076 (0.016)
0.012 (0.015)	0.014 (0.007)	0.002 (0.009)	0.113 (9,995)	-0.002 (0.006)	-0.000 (0.004)	-0.008 (0.007)	-0.0074 (0.007)

Note: Maximum-likelihood estimates with standard errors in parentheses.
Values without reported standard errors are restricted in estimation.

Table 4: Parameter Estimates of the CVAR model

$\mu' \ (\times 10^{-4})$				$\mu^{\mathbb{Q}'} \ (\times 10^{-4})$			
-0.637 (0.448)	-0.215 (0.119)	-0.122 (0.425)	-0.241 (0.095)	0.905 (0.001)	0.111 (0.000)	0.013 (0.003)	0.076 (0.002)
α'				β'			
-0.016 (0.008)	0	0.017 (0.009)	0.009 (0.003)	1	0	-3.097 (0.911)	2.174 (0.715)
0.050 (0.027)	0	-0.033 (0.017)	0	-1	1	0	0
Γ_1				Γ_2			
-0.427 (0.076)	0.183 (0.088)	0.014 (0.060)	-0.064 (0.161)	-0.125 (0.082)	0.164 (0.080)	0.035 (0.065)	0.527 (0.165)
-0.028 (0.032)	0.034 (0.059)	0.148 (0.054)	0.069 (0.100)	-0.025 (0.039)	-0.117 (0.053)	0.057 (0.048)	0.005 (0.114)
0.020 (0.030)	0.024 (0.056)	0.449 (0.077)	0.029 (0.140)	0.015 (0.033)	-0.035 (0.052)	-0.124 (0.079)	0.480 (0.137)
0.020 (0.015)	0.021 (0.027)	0.065 (0.023)	0.236 (0.049)	0.010 (0.016)	-0.009 (0.025)	0.017 (0.033)	0.193 (0.049)
Γ_3				$\Phi^{\mathbb{Q}}$			
0.030 (0.074)	-0.064 (0.083)	-0.190 (0.062)	-0.070 (0.141)	0.905 (0.001)	0.111 (0.000)	0.013 (0.003)	0.076 (0.002)
-0.026 (0.034)	0.056 (0.061)	-0.078 (0.044)	-0.063 (0.099)	-0.010 (0.000)	1.009 (0.000)	0.001 (0.003)	-0.002 (0.002)
0.009 (0.030)	0.066 (0.056)	0.063 (0.070)	0.050 (0.111)	0.156 (0.000)	-0.104 (0.000)	0.879 (0.002)	-0.145 (0.001)
-0.035 (0.014)	-0.017 (0.024)	0.032 (0.028)	0.278 (0.053)	0.168 (0.001)	-0.168 (0.000)	-0.089 (0.003)	0.783 (0.002)
$\Sigma_0(\times 10^{-3})$							
0.369 (0.022)							
0.029 (0.015)	0.228 (0.010)						
0.020 (0.013)	-0.007 (0.016)	0.275 (0.016)					
0.013 (0.005)	0.017 (0.005)	-0.001 (0.008)	0.114 (0.005)				

Note: Maximum-likelihood estimates with standard errors in parentheses.
Values without reported standard errors are restricted in estimation.

Table 5: Parameter Estimates of the Stationary VAR model

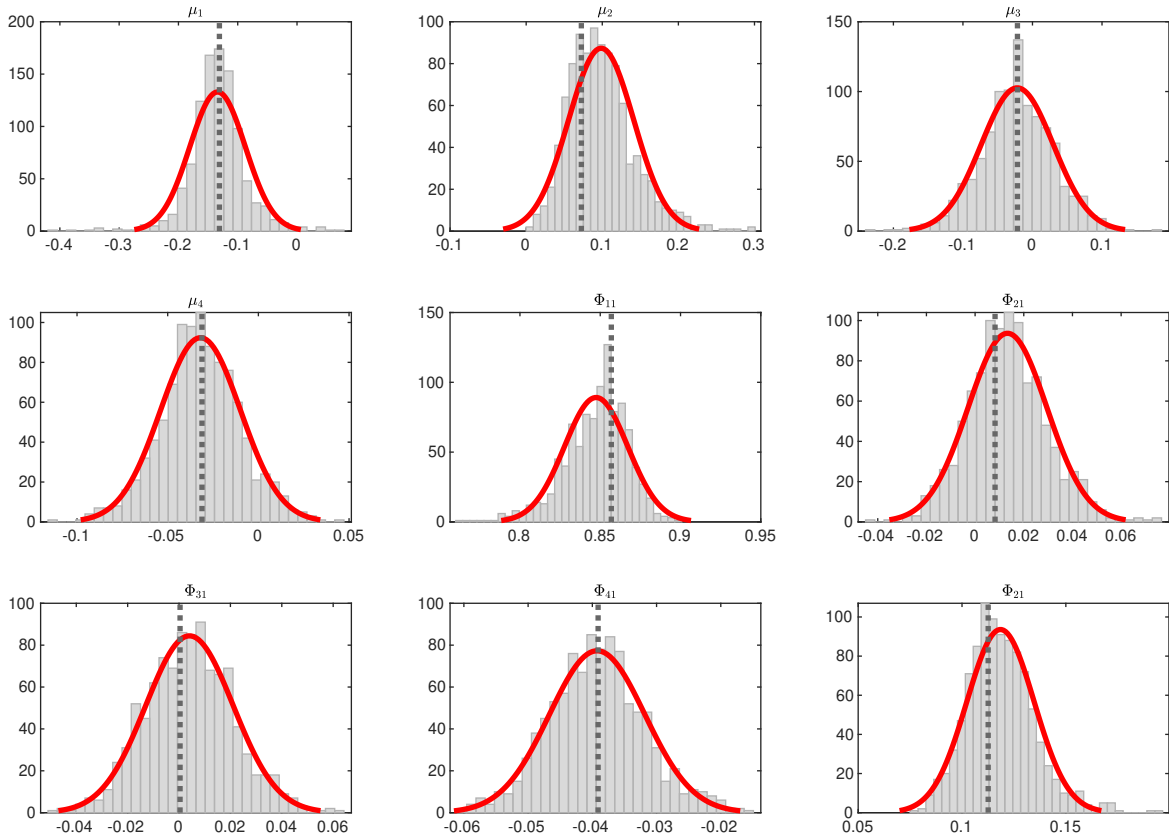
$\mu' (\times 10^{-4})$				$\mu^{\mathbb{Q}'} (\times 10^{-4})$			
-0.608 (0.625)	0.808 (0.119)	-0.265 (0.469)	-0.286 (0.198)	-1.002 (0.584)	-0.116 (0.015)	-2.918 (3.366)	-0.229 (2.619)
Φ_1				Φ_2			
-0.483 (0.052)	0.230 (0.085)	0.047 (0.070)	-0.039 (0.160)	0.291 (0.057)	-0.010 (0.119)	-0.023 (0.119)	0.578 (0.254)
-0.025 (0.032)	1.012 (0.053)	0.149 (0.044)	0.091 (0.100)	0.004 (0.036)	-0.148 (0.074)	-0.091 (0.074)	-0.073 (0.159)
0.056 (0.039)	-0.003 (0.064)	1.384 (0.052)	-0.012 (0.120)	-0.011 (0.043)	-0.055 (0.089)	-0.572 (0.089)	0.444 (0.191)
0.025 (0.017)	0.022 (0.027)	0.040 (0.022)	1.211 (0.051)	-0.013 (0.018)	-0.028 (0.038)	-0.049 (0.038)	-0.045 (0.081)
Φ_3				Φ_4			
0.155 (0.056)	-0.217 (0.117)	-0.215 (0.120)	-0.587 (0.257)	-0.022 (0.051)	0.063 (0.083)	0.159 (0.072)	0.130 (0.161)
0.002 (0.035)	0.168 (0.073)	-0.135 (0.075)	-0.085 (0.160)	0.034 (0.032)	-0.063 (0.052)	0.080 (0.045)	0.074 (0.101)
-0.006 (0.042)	0.109 (0.088)	0.193 (0.090)	-0.420 (0.193)	-0.003 (0.038)	-0.065 (0.062)	-0.081 (0.054)	-0.023 (0.121)
-0.046 (0.018)	-0.006 (0.037)	0.016 (0.038)	0.085 (0.082)	0.033 (0.016)	0.019 (0.026)	-0.034 (0.023)	-0.265 (0.051)
$\Phi^{\mathbb{Q}}$				$\Sigma_0 (\times 10^{-3})$			
0.905 (0.057)	0.111 (0.056)	0.013 (0.030)	0.076 (0.060)	0.364 (0.021)			
-0.010 (0.001)	1.009 (0.001)	0.001 (0.001)	-0.002 (0.001)	0.028 (0.015)	0.226 (0.009)		
0.155 (0.301)	-0.104 (0.297)	0.880 (0.159)	-0.145 (0.316)	0.017 (0.011)	-0.006 (0.017)	0.275 (0.016)	
0.168 (0.245)	-0.168 (0.242)	-0.089 (0.129)	0.783 (0.259)	0.013 (0.005)	0.018 (0.005)	-0.001 (0.006)	0.114 (0.005)

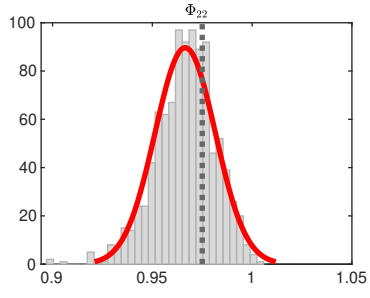
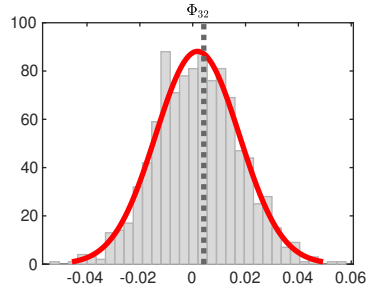
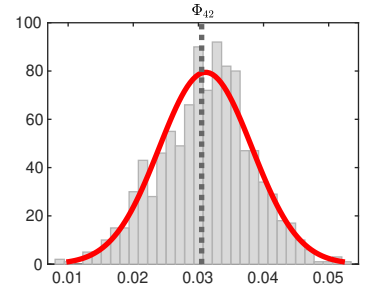
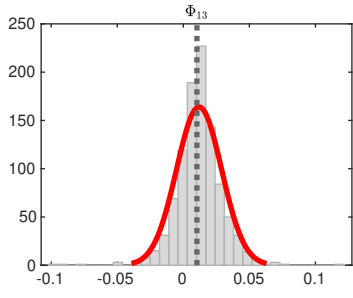
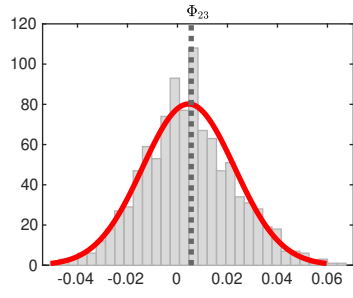
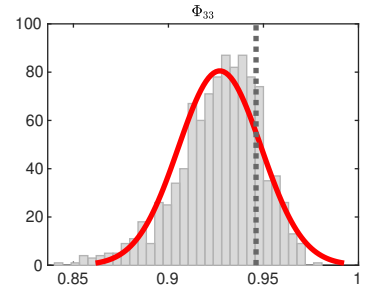
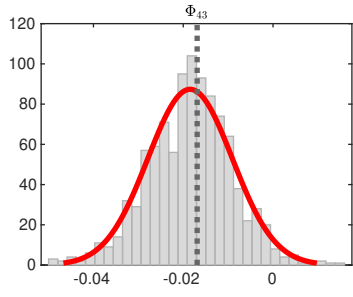
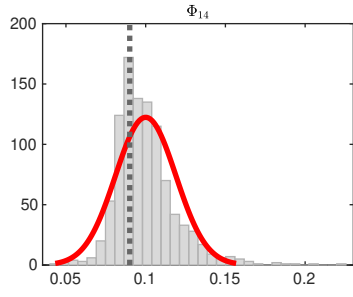
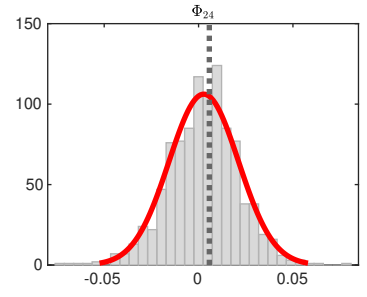
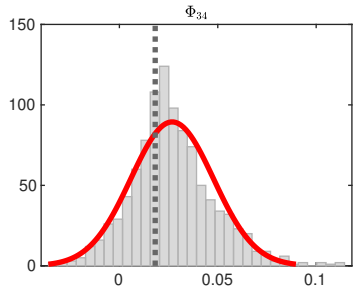
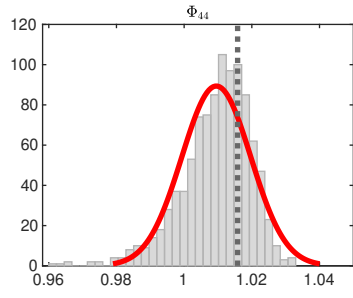
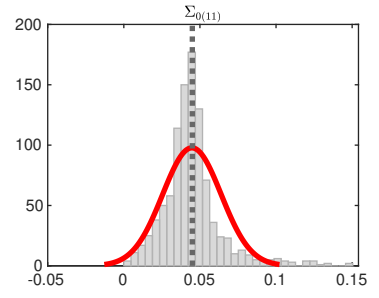
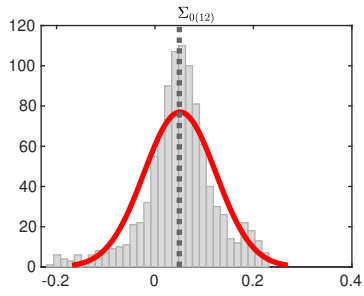
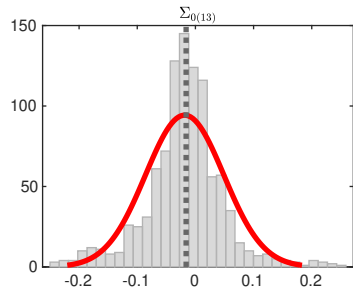
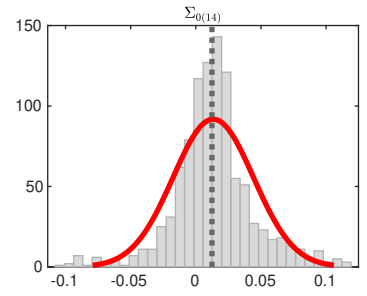
Note: Maximum-likelihood estimates with standard errors in parentheses.

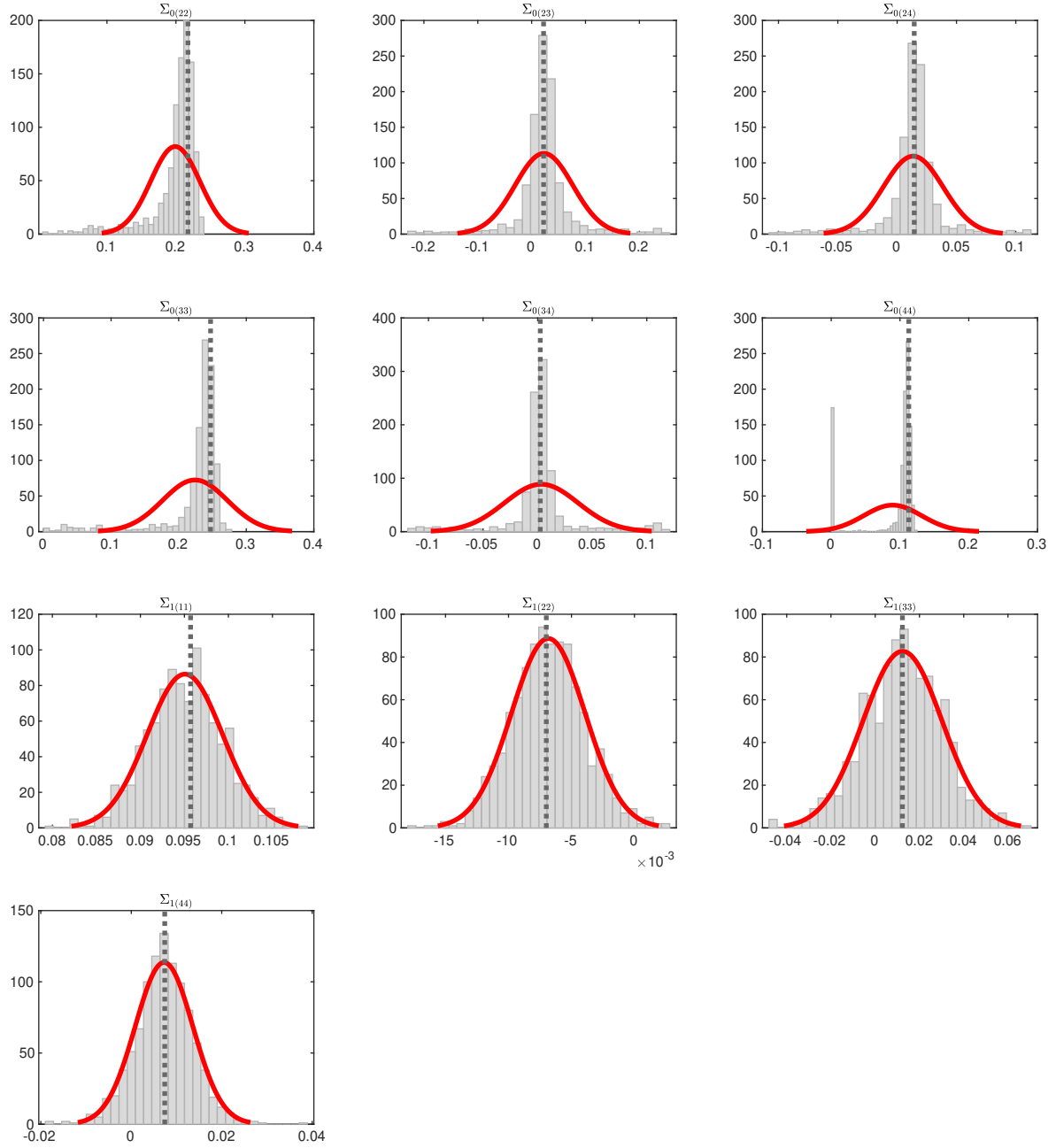
D Simulated Distributions of Maximum Likelihood Estimators

This appendix shows the simulated distributions of maximum likelihood estimates of the $p = 4$ dimensional DAR model in (3) in the main paper with Σ_1 restricted to a diagonal matrix. Parameters are chosen to match the data of the paper. We simulate 1000 paths from this model with a sample length corresponding to the data used in the paper, i.e., $T = 384$. Figure 2 shows the distributions of estimated parameters along with the fitted Gaussian curve (red curve). The parameters of the data generating process are shown with vertical dotted lines. All distributions are approximately Gaussian and centered around the true values with the exception of those related to Σ_0 , which display a large degree of excess kurtosis.

Figure 2: Simulated Distributions of Estimated Parameters







Note: Simulated distributions of the estimated parameters based on 1000 replications with sample length $T = 384$. Red curves represent Gaussian distributions with matching mean and variance, and vertical dotted lines represent parameters of the data generating process.

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