

Online Appendix to “A Joint Model for the Term Structures of Interest Rates and Realized Volatility”

Main paper is available [here](#)

These appendices show results from the robustness check using yield curve data from [Liu and Wu \(2021\)](#) (LW) in contrast to the [Gürkaynak, Sack, and Wright \(2007\)](#) (GSW) data used for the analysis presented in the main paper. Appendix [A](#) shows estimation results and empirical performance of the model, and Appendix [B](#) revisits the discussion on the role of conditional volatility in bond pricing.

A Estimation Results and Performance

The model is estimated using the same specification as presented in the main paper so that only the data are different. Table [1](#) shows the parameter estimates. All state variables are highly persistent with autoregressive coefficients between 0.972 and 0.999. The conditional variance is also persistent with the maximum eigenvalue of $B \otimes B$ equal to 0.56. These coefficients are similar to those estimated using GSW data.

The filtered state variables and their conditional covariation are shown in Figure [1](#). These look similar to those estimated using GSW data when disregarding the scale.

Table [2](#) shows the in-sample performance of the model. The average RMSE for yield levels is 4.830 bps, which is slightly higher than for the GSW data. Compared to the main results, the model matches the 1- to 7-year LW yields with less precision, but the 10-year LW yield with higher precision. Realized covariation is matched with precision almost identical to the results obtained using GSW data. Thus, Figure [2](#), showing realized against model-implied conditional variances and covariances, is also very similar to the corresponding figure in the main paper.

In sum, results regarding empirical performance of the model do not change notably when switching from GSW to LW data.

Table 1: Parameter Estimates

	Estimate	Stddev		Estimate	Stddev
δ_0	7.932	(0.069)	$\Phi_{1,3}^Q$	0.008	(0.004)
$\Phi_{1,1}$	0.999	(0.002)	$\Phi_{2,1}^Q$	-0.232	(0.049)
$\Phi_{2,2}$	0.972	(0.027)	$\Phi_{2,2}^Q$	0.965	(0.010)
$\Phi_{3,3}$	0.999	(0.003)	$\Phi_{2,3}^Q$	-0.057	(0.031)
$C_{1,1}$	0.000	(0.000)	$\Phi_{3,1}^Q$	-0.033	(0.029)
$C_{2,1}$	0.015	(0.010)	$\Phi_{3,2}^Q$	-0.014	(0.006)
$C_{2,2}$	0.085	(0.019)	$\Phi_{3,3}^Q$	0.982	(0.010)
$C_{3,1}$	0.000	(0.011)	R_1^Y	0.012	(0.005)
$C_{3,2}$	0.001	(0.001)	R_2^Y	0.000	(0.000)
$C_{3,3}$	0.002	(0.021)	R_3^Y	0.002	(0.003)
$A_{1,1}$	0.164	(0.035)	R_4^Y	0.001	(0.006)
$A_{2,2}$	0.032	(0.006)	R_5^Y	0.001	(0.003)
$B_{1,1}$	0.639	(0.045)	R_6^Y	0.001	(0.012)
$B_{2,2}$	0.748	(0.055)	R_1^{RV}	0.000	(0.001)
μ_1^Q	0.017	(0.010)	R_2^{RV}	0.000	(0.000)
μ_2^Q	0.005	(0.003)	R_3^{RV}	0.000	(0.000)
μ_3^Q	0.016	(0.005)	R_4^{RV}	0.000	(0.000)
$\Phi_{1,1}^Q$	1.003	(0.003)	R_5^{RV}	0.000	(0.000)
$\Phi_{1,2}^Q$	0.005	(0.001)	R_6^{RV}	0.000	(0.001)

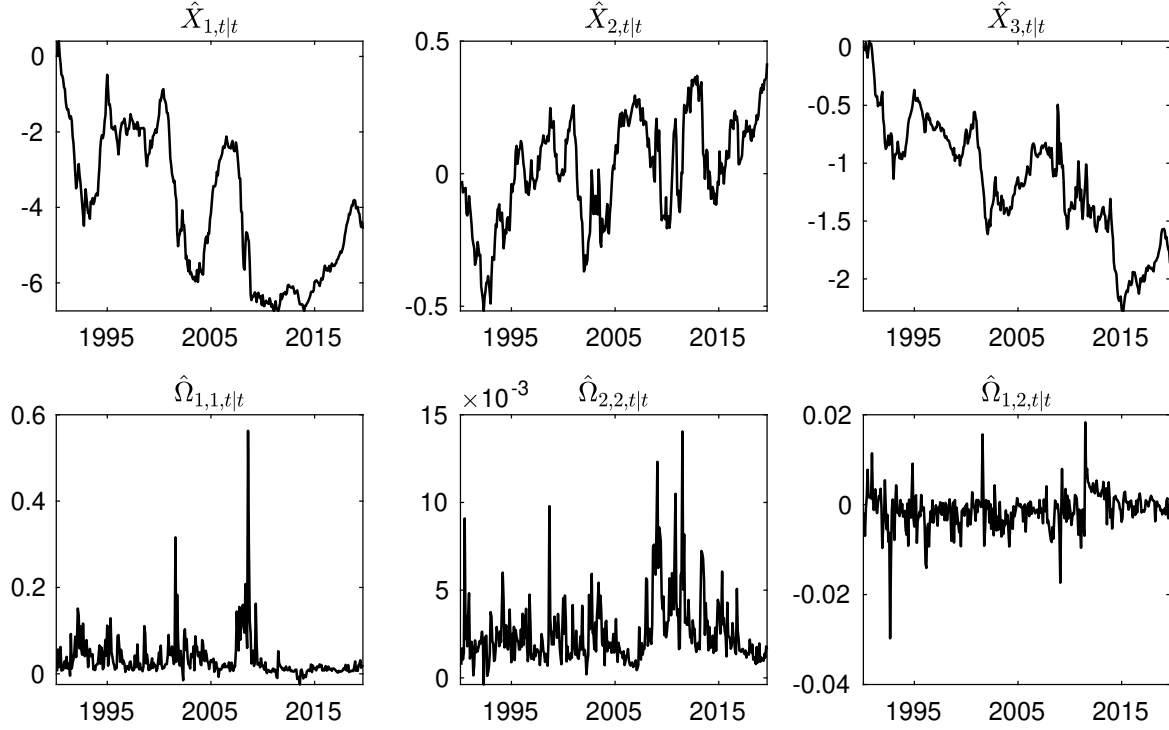
Notes: The table shows parameter estimates and standard deviations computed by the Huber sandwich estimator.

B The Role of Conditional Covariation in Bond Pricing

Similar to the baseline results, the model prices conditional covariation in long-maturity bonds. Table 3 shows the RMSEs when shutting down the channel of time-varying conditional covariation. This increases the RMSE of the 10-year yield from 1.161 to 12.406 bps, and the average RMSE from 4.830 to 7.566 bps. These differences are similar to those reported in the main paper using GSW data.

The remainder of this appendix discusses the source of the role of conditional covariation in bond pricing. Similar to the baseline results, Q-convexity is negligible in the LW data set. The remaining channel is therefore the filtered state variables.

Figure 1: Filtered State Vector and Conditional Covariance Matrix



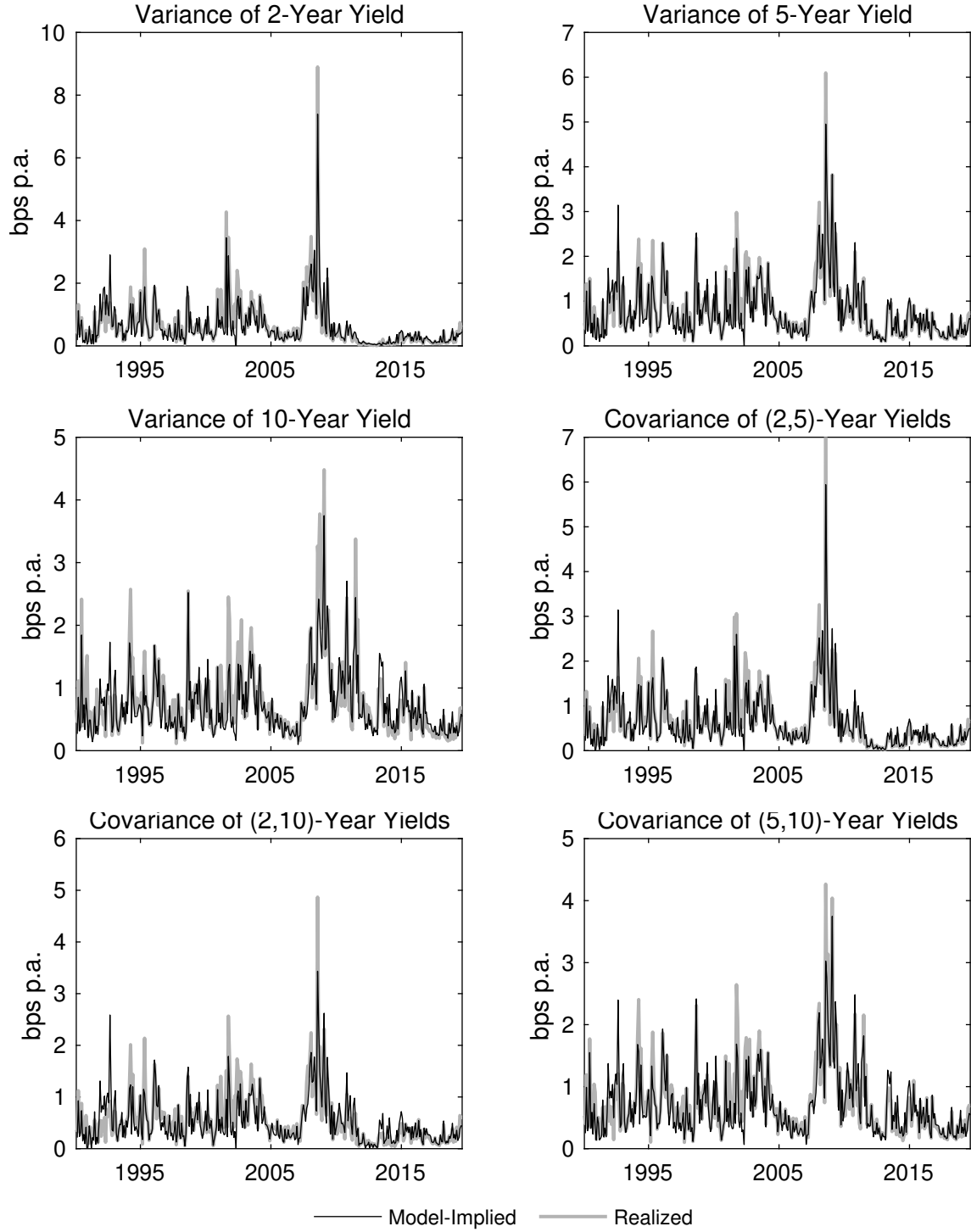
Notes: The figure shows the filtered state vector $\hat{X}_{t|t}$ and the time-varying variances and covariances in $\hat{\Omega}_{t|t}$.

Table 2: In-Sample Performance

	Yield Levels						
	1-year	2-year	3-year	5-year	7-year	10-year	Average
Model-Implied	10.583	0.000	3.785	1.326	3.044	1.161	4.830
	Realized Yield Variances			Realized Yield Covariances			
	2-year	5-year	10-year	(2,5)-year	(2,10)-year	(5,10)-year	
Model-Implied	0.285	0.207	0.288	0.247	0.264	0.242	
Sample Average	0.751	0.664	0.593	0.657	0.489	0.568	

Notes: The upper panel shows RMSEs of model-implied yields. The lower panel shows RMSEs of model-implied conditional variances and covariances. These are compared to RMSEs constructed using the sample mean of realized covariation. All numbers are reported in bps p.a.

Figure 2: Realized and Model-Implied Conditional Covariation



Notes: The figure shows model-implied conditional against realized covariation.

Table 3: In-Sample Performance with $A = B = 0_{p \times p}$

	Yield Levels						
	1-year	2-year	3-year	5-year	7-year	10-year	Average
Maturity							
Model-Implied	10.583	0.000	3.785	1.326	3.044	1.161	4.830
$A = B = 0_{p \times p}$	10.578	0.405	3.890	3.323	7.163	12.406	7.566

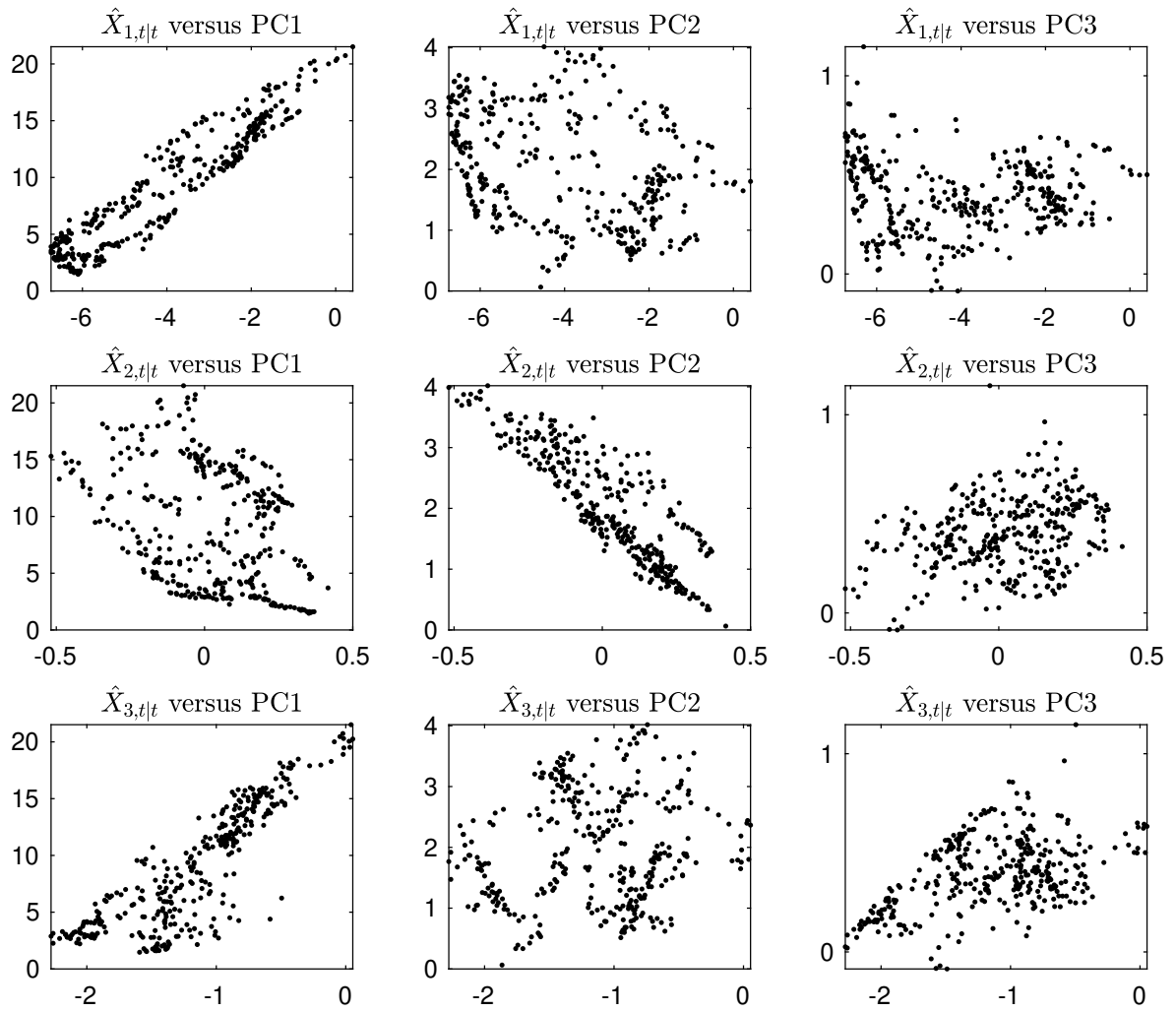
Notes: The table shows RMSEs in bps of model-implied yields along with RMSEs without time-varying conditional covariation, i.e., when $A = B = 0_{p \times p}$ in the recursions associated with the bond pricing equation.

Figure 3 compares the filtered state variables to the first three principal components of the LW yield curve data. As concluded in the main paper, the first and third state variables are mainly associated with the first principal component, while the second state variable is related to the second principal component. As discussed in the main paper, these findings contrast with those of affine term structure models with constant conditional covariation.

Finally, the bond pricing loadings on the state variables and their conditional covariation are shown in Figure 4. The loadings are very similar to those reported in the main paper, up to scale.

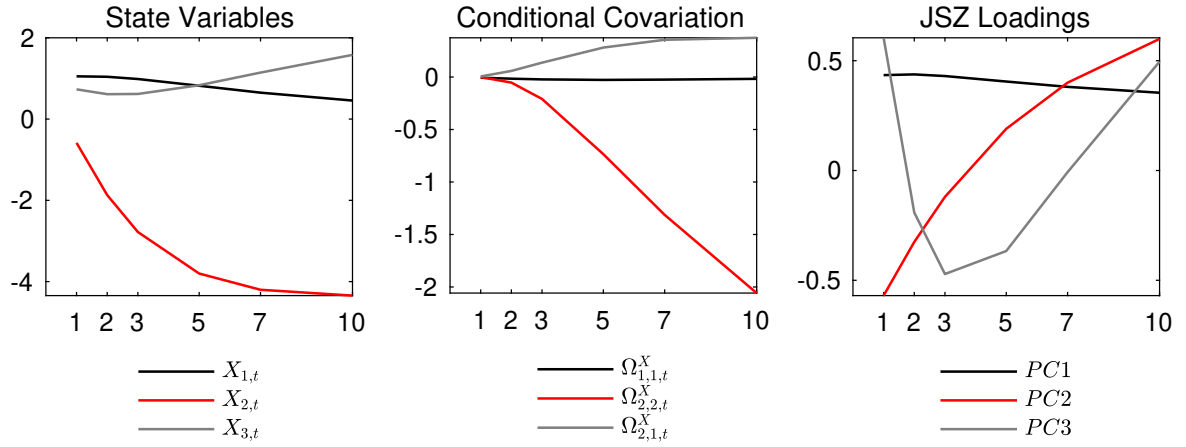
In conclusion, the results are not materially different when using the LW data set instead of the GSW data to estimate the model.

Figure 3: Filtered State Variables Versus Yield Curve Principal Components



Notes: The figure shows scatter plots of the filtered state vector $\hat{X}_{t|t}$ against the first three principal components of the yield data.

Figure 4: Loadings



Notes: The figure shows model-implied bond yield loadings on X_t in the left-hand chart, loadings on Ω_t^X in the middle chart, and loadings on the three first principal components in the Gaussian term structure model (Joslin, Singleton, and Zhu, 2011) in the right-hand panel.

Bibliography

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